

Smooth densities for SDEs driven by degenerate subordinated Brownian motion with state-dependent switching *

Xiaobin Sun^{a†} Yingchao Xie^{b‡}

a. School of Mathematical Sciences, Nankai University, Tianjin 300071, China.

b. School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, China.

Abstract. In this paper, by using Malliavin calculus, under uniform Hörmander's type condition, we obtain the existence of smooth densities for the solutions of stochastic differential equations driven by degenerate subordinated Brownian motion with state-dependent switching.

Keywords: Malliavin calculus, state-dependent switching, subordinated Brownian motion, α -stable process, smoothness of density

Mathematics Subject Classification (2000). Primary: 34D08, 34D25; Secondary: 60H20

1 Introduction

This paper considers the following jump-diffusion with state-dependent switching on \mathbb{R}^n :

$$dX_t = b(X_t, \alpha_t)dt + \sigma dL_t, \quad (X_0, \alpha_0) = (x, \alpha) \in \mathbb{R}^n \times \mathbb{S}, \quad (1.1)$$

where $b : \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^n$ is an appropriate function, σ is a $n \times d$ constant matrix, L_t is a d -dimensional subordinated Brownian motion, $\mathbb{S} = \{1, 2, \dots, m_0\}$, and $\{\alpha_t, t \geq 0\}$ is an \mathbb{S} -valued state-dependent switching process described by

$$\mathbb{P}\{\alpha_{t+\Delta} = j | \alpha_t = i, X_s, \alpha_s, s \leq t\} = \begin{cases} q_{ij}(X_t)\Delta + o(\Delta), & i \neq j \\ 1 + q_{ii}(X_t)\Delta + o(\Delta), & i = j. \end{cases} \quad (1.2)$$

Stochastic differential equations with state-dependent switching have been studied in the literature and they have applications to mathematical finance. Many important problems related to this kind of SDEs, such as existence and uniqueness of solutions, Feller property, strong Feller property, ergodicity and etc, have been studied (see [9, 10, 12]). However, the smoothness of the densities of the solutions to this kind of SDEs has not been studied much. If the noise is the classical Brownian motion, the smoothness of the densities of the solutions has been proved under the uniform Hörmander's condition in [4]. Smoothness of densities of solutions for SDEs driven by subordinated Brownian motions has been studied in

*Research is supported by NSFs of China (No.11271169) and the Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

[†]E-mail: xbsun@mail.nankai.edu.cn

[‡]E-mail: ycxie@jsnu.edu.cn

[5, 13, 14]. In [5], Kusuoka proved the solution has a smooth density under a nondegenerate condition on σ . Using Malliavin calculus, Zhang established in [13], the solution has smooth density under a special degenerate case. Furthermore, Zhang improved this result in [14], i.e., the solution has smoothness of the density in the uniform Hörmander's type condition. Similarly, Xie used Malliavin calculus to obtain the estimations of the smooth densities of one-dimension stochastic differential equation with jumps in [11].

In this paper we add the switching term into the SDE driven by degenerate subordinated Brownian motions, i.e., the solution (X_t, α_t) satisfies equations (1.1) and (1.2). For the reason of technique, we only consider the case of additive noise in this paper. We remark that the results of this paper can be applied to case of the noise is an α -stable process. In order to show the smoothness of density for X_t , we need to develop Malliavin calculus for X_t . The difficulty here is the appearance of the switching term α_t . Our procedure is to follow the method in [4], i.e., we consider the product probability space produced by a Brownian motion, a subordinator and a Poisson random measure. We will perturb the Brownian motion, while keeping the other two quantities unchanged.

As is well-known, the most difficult part in applying Malliavin calculus to study the smoothness of the density is to prove that the Malliavin covariance matrix has all the negative moments. When the switching is not present, this was done in [14] in the uniform Hörmander's type condition. When switching is present, the fact that X_t depends on the jump process α_t makes things even more difficult. We will find some random intervals on which α_t is fixed, and then follow the approach developed in [14] under these intervals. More precisely, we will use the following strategy (see [3, 4]): First we notice that the jump times of α_t form a subset of the jump times of some Poisson process N_t , independent of the subordinated Brownian motion. Then conditioning on $N_t = k$, there exists some random interval $[T_1, T_2)$ with $0 \leq T_1 < T_2 \leq t$ such that $T_2 - T_1 \geq \frac{t}{k+1}$ and $\alpha_t = \alpha_{T_1}, t \in [T_1, T_2)$. On this time interval, we apply the Norris' type lemma developed in [14].

The paper is organized as follows. In the next section, we introduce some notation and assumptions that we use throughout the paper. We develop the Malliavin calculus for X_t in Section 3. In Section 4, we consider the small jump and the large jump separately. We first show that the Malliavin covariance matrix has all negative finite moments under the uniform Hörmander's type condition, assuming subordinator S_t has finite moments of all orders, then we prove that X_t has smooth density. Finally, using the techniques of subsection 3.3 in [13], we also prove X_t has smooth density without the assumption of S_t has finite moments of all orders.

2 Preliminaries

Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ be the d -dimensional canonical Wiener space. That is, Ω_1 is the set of all continuous maps $\omega_1 : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ such that $\omega_1(0) = 0$ and \mathbb{P}_1 is the canonical Wiener measure such that coordinate process

$$W_t(\omega_1) := \omega_1(t)$$

is a standard d -dimensional Brownian motion.

Let $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ be the space of all increasing, purely discontinuous and càdlàg functions from \mathbb{R}_+ to \mathbb{R}_+ with $\omega_2(0) = 0$, which is endowed with the Skorohod metric and the probability measure \mathbb{P}_2 so that the coordinate process

$$S_t(\omega_2) := \omega_2(t)$$

is an increasing one dimensional Lévy process (called a subordinator) on \mathbb{R}_+ with Laplace transform:

$$\mathbb{E}_2 e^{-sS_t} = \exp \left\{ t \int_0^\infty (e^{-su} - 1) \nu_S(du) \right\},$$

where \mathbb{E}_2 is the expectation with respect to \mathbb{P}_2 , ν_S is the Lévy measure satisfying $\nu_S(\{0\}) = 0$ and

$$\nu_S((-\infty, 0]) = 0, \quad \int_0^\infty (1 \wedge u) \nu_S(du) < \infty.$$

Let $(\Omega_3, \mathcal{F}_3, \mathbb{P}_3)$ be a complete probability space, on which $N(dt, dz)$ is a Poisson random measure with intensity $dt\lambda(dz)$, where $\lambda(\cdot)$ is the Lebsgue measure on \mathbb{R} .

We will use $(\Omega, \mathcal{F}, \mathbb{P})$ to denote the product probability space $(\Omega_1 \times \Omega_2 \times \Omega_3, \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3, \mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3)$. We extend W_t , S_t and $N(dt, dz)$ to Ω by letting $W_t(\omega) = \omega_1(t)$, $S_t(\omega) = \omega_2(t)$ and $N(dt, dz, \omega) = N(dt, dz, \omega_3)$ respectively. Thus on $(\Omega, \mathcal{F}, \mathbb{P})$, W_t , S_t and $N(dt, dz)$ are independent. We define

$$L_t(\omega) := W_{S_t}(\omega) = \omega_1(\omega_2(t)).$$

Then $(L_t)_{t \geq 0}$ is a Lévy process (called a subordinated Brownian motion) with characteristic function:

$$\mathbb{E} e^{iz \cdot L_t} = \exp \left\{ t \int_{\mathbb{R}^d} (e^{iz \cdot y} - 1 - iz \cdot y 1_{|y| \leq 1}) \nu_L(dy) \right\},$$

where \mathbb{E} is the expectation with respect to \mathbb{P} , ν_L is the Lévy measure given by

$$\nu_L(\Gamma) = \int_0^\infty (2\pi s)^{-d/2} \left(\int_\Gamma e^{-\frac{|y|^2}{2s}} dy \right) \nu_S(ds),$$

where $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. Obviously, ν_L is a symmetric measure.

Let $\mathbb{S} = \{1, 2, \dots, m_0\}$, where m_0 is a given positive integer which will be fixed throughout the paper. For $k \in \mathbb{N}$ we denote by $C^k(\mathbb{R}^n \times \mathbb{S}; \mathbb{R}^n)$ the family of all \mathbb{R}^n -valued functions $f(x, \alpha)$ on $\mathbb{R}^n \times \mathbb{S}$ which are k -times continuously differentiable in x for any $\alpha \in \mathbb{S}$. The k -th derivative tensor of f with respect to x is denoted by $\nabla^k f(x, \alpha)$.

For $x \in \mathbb{R}^n$ and $\sigma \in \mathbb{R}^n \times \mathbb{R}^d$, we use the notation $|x|^2 = \sum_{i=1}^n |x_i|^2$ and $|\sigma|^2 = \sum_{i=1}^n \sum_{j=1}^d |\sigma_{ij}|^2$. We will consider the metric Λ on $\mathbb{R}^n \times \mathbb{S}$ given by $\Lambda((x, i), (y, j)) = |x - y| + d(i, j)$, for $x, y \in \mathbb{R}^n, i, j \in \mathbb{S}$, where $d(i, j) = 0$ if $i = j$ and $d(i, j) = 1$ if $i \neq j$. Let $\mathcal{B}_b(\mathbb{R}^n \times \mathbb{S})$ be the family of all bounded Borel measurable functions on $\mathbb{R}^n \times \mathbb{S}$.

Now we make the following assumptions on function $b : \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^n$.

(H1) There is a positive constant C_1 such that

$$|b(x, i) - b(y, i)| \leq C_1 |x - y| \quad \text{for any } x, y \in \mathbb{R}^n, i \in \mathbb{S}.$$

(H2) For each $i \in \mathbb{S}$, the function $b(\cdot, i)$ belongs to $C^2(\mathbb{R}^n; \mathbb{R}^n)$ and has first and second bounded partial derivatives.

It is clear that **(H2)** implies **(H1)**.

The switching rate $Q(x) = (q_{ij}(x))$ is assumed to satisfy the following assumption.

(i) $q_{ij}(x)$ is Borel measurable and there exists $K > 0$ such that

$$\sup_{i, j \in \mathbb{S}, x \in \mathbb{R}^n} |q_{ij}(x)| \leq K,$$

- (ii) $q_{ij}(x) \geq 0$ for $x \in \mathbb{R}^n$ and $i \neq j$,
- (iii) $q_{ii}(x) = -\sum_{j \neq i} q_{ij}(x)$ for $x \in \mathbb{R}^n$ and $i \in \mathbb{S}$.

It is well known (see [9]) that the process $\{\alpha_t, 0 \leq t \leq T\}$ can be described as follows. Introduce a function $g : \mathbb{R}^n \times \mathbb{S} \times [0, m_0(m_0 - 1)K] \rightarrow \mathbb{R}$ defined by

$$g(x, i, z) = \sum_{j \in \mathbb{S} \setminus \{i\}} (j - i) 1_{z \in \Delta_{ij}(x)}, \quad \forall i \in \mathbb{S},$$

where $\Delta_{ij}(x)$'s are the consecutive (with respect to the lexicographic ordering on $\mathbb{S} \times \mathbb{S}$) left-closed, right-open intervals of \mathbb{R}_+ , each having length $q_{ij}(x)$, with $\Delta_{12}(x) = [0, q_{12}(x))$. Then, (1.2) can be rewritten as

$$\alpha(t) = \alpha + \int_0^t \int_{[0, m_0(m_0-1)K]} g(X_{s-}, \alpha_{s-}, z) N(ds, dz). \quad (2.1)$$

Using the method in [1] or [2], we have the following:

Theorem 2.1 *Suppose that the condition **(H1)** holds. For any given initial value $(x, \alpha) \in \mathbb{R}^n \times \mathbb{S}$, there exists a unique solution $\{(X_t, \alpha_t), t \geq 0\}$ to equations (1.1) and (2.1).*

If we consider the natural filtration:

$$\mathcal{F}_t := \sigma\{L_t, N([0, t], \Gamma) : t \geq 0, \Gamma \in \mathcal{B}([0, m_0(m_0 - 1)K])\},$$

then the solution (X_t, α_t) is a Markov process and the associated Markov semigroup P_t satisfies

$$P_t f(x, \alpha) = \mathbb{E} f(X_t(x), \alpha_t(\alpha)), \quad t > 0, f \in \mathcal{B}_b(\mathbb{R}^n \times \mathbb{S}).$$

3 The Malliavin calculus for SDE (1.1)

In this section we analyze the regularity, in the sense of Malliavin calculus, of the random vector X_t solution to the system (1.1) and (2.1). Denote by H the Hilbert space $H = L^2([0, \infty); \mathbb{R}^d)$, equipped with the inner product $\langle h_1, h_2 \rangle_H = \int_0^\infty \langle h_1(s), h_2(s) \rangle_{\mathbb{R}^d} ds$.

For a Hilbert space U and a real number $p \geq 1$, we denote by $L^p(\Omega_1; U)$ the space of U -valued random variables ξ such that $\mathbb{E}_1 \|\xi\|_U^p < \infty$, where \mathbb{E}_1 is the expectation in the probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$. We also set $L^{\infty-}(\Omega_1; U) := \cap_{p < \infty} L^p(\Omega_1; U)$.

We introduce the derivative operator for a random variable F in the space $L^{\infty-}(\Omega_1; U)$ following the approach of Malliavin in [6]. We say that F belongs to $\mathbb{D}^{1,\infty}(U)$ if there exists $DF \in L^{\infty-}(\Omega_1; H \otimes U)$ such that for any $h \in H$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_1 \left\| \frac{F(\omega_1 + \varepsilon \int_0^\cdot h_s ds) - F(\omega_1)}{\varepsilon} - \langle DF, h \rangle_H \right\|_U^p = 0$$

holds for every $p \geq 1$. In this case, we define the Malliavin derivative of F in the direction h by $D^h F := \langle DF, h \rangle_H$. Then, for any $p \geq 1$ we define the Sobolev space $\mathbb{D}^{1,p}(U)$ as the completion of $\mathbb{D}^{1,\infty}(U)$ under the following norm

$$\|F\|_{1,p,U} = [\mathbb{E}_1(\|F\|_U^p)]^{1/p} + [\mathbb{E}_1(\|DF\|_{H \otimes U}^p)]^{1/p}.$$

By induction we define the k th derivative by $D^k F = D(D^{k-1} F)$, which is a random element with values in $H^{\otimes k} \otimes U$. For any integer $k \geq 1$, the Sobolev space $\mathbb{D}^{k,p}(U)$ is the completion of $\mathbb{D}^{k,\infty}(U)$ under the norm

$$\|F\|_{k,p,U} = \|F\|_{k-1,p,U} + \|D^k F\|_{1,p,H^{\otimes k} \otimes U}.$$

It turns out that D is a closed operator from $L^p(\Omega_1; U)$ to $L^p(\Omega_1; H \otimes U)$. Its adjoint δ is called the divergence operator, and is continuous from $L^p(\Omega_1; H \otimes U)$ to $L^p(\Omega_1; U)$ for any $p > 1$. The duality relationship reads

$$\mathbb{E}_1(\langle DF, u \rangle_{H \otimes U}) = \mathbb{E}_1(\langle F, \delta(u) \rangle_U),$$

for any $F \in \mathbb{D}^{1,2}(U)$ and $u \in \mathcal{D}(\delta)$ which is the domain of δ .

A square integrable random variable $F \in L^2(\Omega)$ can be identified with an element of $L^2(\Omega_1; V)$, where $V = L^2(\Omega_2 \times \Omega_3)$.

We assume that S_t has finite moments (i.e. $\mathbb{E}|S_t|^p < \infty$, for all $p \geq 1, t > 0$) in this section.

The following theorem is the main result of this section. The key idea of the proof of following theorem is inspired from Theorem 4.2 in [12], in which Yin and Zhu study the differentiability in mean square with respect to initial value x for $X_t(x)$, and as known to all, the Malliavin derivative actually is the derivative with respect to path space. Hence, we believe that the techniques used in Theorem 4.2 in [12] also can be applied to study the Mallivian differentiability.

Theorem 3.1 *Suppose that condition (H2) holds. Then for any $t > 0$, $h \in H$, $X_t \in \mathbb{D}^{1,\infty}(\mathbb{R}^n \otimes V)$ and $D^h X_t$ satisfies*

$$\begin{cases} dD^h X_t = \nabla b(X_t, \alpha_t) D^h X_t dt + \sigma d \left(\int_0^{S_t} h_s ds \right), \\ D^h X_0 = 0. \end{cases} \quad (3.1)$$

Let $(X_t^{\varepsilon h}(x), \alpha_t^{\varepsilon h}(\alpha))$ be the solution of (1.1) and (1.2) with W_{S_t} replaced by $W_{S_t} + \varepsilon \int_0^{S_t} h_s ds$, i.e.,

$$\begin{cases} dX_t^{\varepsilon h} = b(X_t^{\varepsilon h}, \alpha_t^{\varepsilon h}) dt + \sigma dW_{S_t} + \varepsilon \sigma d \left(\int_0^{S_t} h_s ds \right), \\ (X_0^{\varepsilon h}, \alpha_0^{\varepsilon h}) = (x, \alpha) \in \mathbb{R}^n \times \mathbb{S} \end{cases} \quad (3.2)$$

and

$$\mathbb{P}\{\alpha_{t+\Delta}^{\varepsilon h} = j | \alpha_t^{\varepsilon h} = i, X_s^{\varepsilon h}, \alpha_s^{\varepsilon h}, s \leq t\} = \begin{cases} q_{ij}(X_t^{\varepsilon h}) \Delta + o(\Delta), & i \neq j \\ 1 + q_{ii}(X_t^{\varepsilon h}) \Delta + o(\Delta), & i = j. \end{cases}$$

Thus, we have

$$\frac{X_t^{\varepsilon h} - X_t}{\varepsilon} = \frac{1}{\varepsilon} \int_0^t [b(X_s^{\varepsilon h}, \alpha_s) - b(X_s, \alpha_s)] ds + \sigma \int_0^{S_t} h_s ds + \phi_t^{\varepsilon h},$$

where

$$\phi_t^{\varepsilon h} = \frac{1}{\varepsilon} \int_0^t [b(X_s^{\varepsilon h}, \alpha_s^{\varepsilon h}) - b(X_s^{\varepsilon h}, \alpha_s)] ds.$$

In order to prove Theorem 3.1, we first give some preliminary lemmas. In the remainder of this paper, C will denote a generic constant which may vary from line to line and it might depend on T , the exponent p , the initial value x and a fixed element $h \in H$.

Lemma 3.1 Suppose that condition **(H1)** holds. Then for any $T > 0$, $h \in H$ and $p \geq 2$, we have

$$\mathbb{E} \left[\sup_{t \leq T} |X_t^{\varepsilon h}|^p \right] \leq C.$$

Proof From (3.2) it is easy to see that

$$\begin{aligned} |X_t^{\varepsilon h}|^p &\leq C \left[|x|^p + \left| \int_0^t b(X_s^{\varepsilon h}, \alpha_s^{\varepsilon h}) ds \right|^p + |\sigma|^p |W_{S_t}|^p + \varepsilon^p \left| \int_0^{S_t} \sigma h_s ds \right|^p \right] \\ &:= C [|x|^p + I_1(t) + I_2(t) + I_3(t)]. \end{aligned}$$

By the condition (H1), Hölder's inequalities and the assumption that S_t has finite moments, we obtain

$$\mathbb{E} \left[\sup_{t \leq T} (I_1(t) + I_2(t) + I_3(t)) \right] \leq C \int_0^T (\mathbb{E} |X_s^{\varepsilon h}|^p + 1) ds.$$

Then the desired estimate follows from Gronwall's lemma. ♣

Lemma 3.2 Suppose that condition **(H1)** holds. Then for any $h \in H$, $p \geq 2$ and $0 \leq s \leq t$ with $|t - s| \leq 1$, we have

$$\mathbb{E} |X_t^{\varepsilon h} - X_s^{\varepsilon h}|^p \leq C(t - s).$$

Proof Note that for $0 \leq s \leq t$ with $|t - s| \leq 1$,

$$X_t^{\varepsilon h} - X_s^{\varepsilon h} = \int_s^t b(X_r^{\varepsilon h}, \alpha_r^{\varepsilon h}) dr + \sigma(W_{S_t} - W_{S_s}) + \varepsilon \int_{S_s}^{S_t} \sigma h_r dr.$$

Then, we have

$$|X_t^{\varepsilon h} - X_s^{\varepsilon h}|^p \leq C \left[\left| \int_s^t b(X_r^{\varepsilon h}, \alpha_r^{\varepsilon h}) dr \right|^p + |\sigma|^p |W_{S_t} - W_{S_s}|^p + \varepsilon^p \left| \int_{S_s}^{S_t} \sigma h_r dr \right|^p \right].$$

Notice that S_t has finite moments, we can obtain $\mathbb{E} |S_t - S_s|^p \leq C|t - s|$. Combining this with Hölder's inequalities and Lemma 3.1 yield

$$\mathbb{E} |X_t^{\varepsilon h} - X_s^{\varepsilon h}|^p \leq C|t - s|.$$

The proof is complete. ♣

The following lemma will be a basic ingredient in the proof of Theorem 3.1. Using Lemmas 3.1, 3.2 and following the steps of Lemma 3.4 in [4] or Lemma 4.3 in [12], we can obtain the following lemma. Because the proof is almost the same, we omit the proof.

Lemma 3.3 Suppose that condition **(H1)** holds. Then for any $T > 0$, $h \in H$ and $p \geq 2$ we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\sup_{t \leq T} |\phi_t^{\varepsilon h}|^p \right] = 0.$$

Lemma 3.4 Suppose that condition **(H1)** holds. Then for any $T > 0$, $h \in H$ and $p \geq 2$, we have

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^{\varepsilon h} - X_t|^p \right] \leq C\varepsilon^p.$$

Proof We write

$$X_t^{\varepsilon h} - X_t = \int_0^t [b(X_s^{\varepsilon h}, \alpha_s^{\varepsilon h}) - b(X_s, \alpha_s)] ds + \varepsilon \int_0^{S_t} \sigma h_s ds := A(t) + B(t),$$

where

$$A(t) = \int_0^t [b(X_s^{\varepsilon h}, \alpha_s^{\varepsilon h}) - b(X_s^{\varepsilon h}, \alpha_s)] ds$$

and

$$B(t) = \int_0^t [b(X_s^{\varepsilon h}, \alpha_s) - b(X_s, \alpha_s)] ds + \varepsilon \int_0^{S_t} \sigma h_s ds.$$

With this notation we can write

$$\begin{aligned} \sup_{0 \leq t \leq T} |X_t^{\varepsilon h} - X_t|^p &\leq C \left(\sup_{0 \leq t \leq T} |A(t)|^p + \sup_{0 \leq t \leq T} |B(t)|^p \right) \\ &= C\varepsilon^p \sup_{0 \leq t \leq T} |\phi_t^{\varepsilon h}|^p + C \sup_{0 \leq t \leq T} |B(t)|^p. \end{aligned}$$

Hölder's inequalities and the assumption that S_t has finite moments yield

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^{\varepsilon h} - X_t|^p \right] \leq C\varepsilon^p \mathbb{E} \left[\sup_{0 \leq t \leq T} |\phi_t^{\varepsilon h}|^p \right] + C \int_0^T \mathbb{E} |X_t^{\varepsilon h} - X_t|^p dt + C\varepsilon^p.$$

By Lemma 3.3 and Gronwall's inequality, we obtain

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t^{\varepsilon h} - X_t|^p \right] \leq C\varepsilon^p,$$

which completes the proof. ♣

The proof of Theorem 3.1:

Let ψ_t^h be the solution of equation (3.1) and it is easy to verify that $\mathbb{E} [\sup_{s \leq t} |\psi_s^h|^p] \leq C$. Then, we have

$$\begin{aligned} &\frac{X_t^{\varepsilon h} - X_t}{\varepsilon} - \psi_t^h \\ &= \frac{1}{\varepsilon} \int_0^t [b(X_s^{\varepsilon h}, \alpha_s) - b(X_s, \alpha_s) - \varepsilon \nabla b(X_s, \alpha_s) \psi_s^h] ds + \phi_t^{\varepsilon h} \\ &= \int_0^t \left(\int_0^1 \nabla b(X_s + \nu(X_s^{\varepsilon h} - X_s), \alpha_s) d\nu \right) \frac{X_s^{\varepsilon h} - X_s}{\varepsilon} - \nabla b(X_s, \alpha_s) \psi_s^h ds + \phi_t^{\varepsilon h} \\ &= \int_0^t \left(\int_0^1 \nabla b(X_s + \nu(X_s^{\varepsilon h} - X_s), \alpha_s) d\nu \right) \left(\frac{X_s^{\varepsilon h} - X_s}{\varepsilon} - \psi_s^h \right) ds + \phi_t^{\varepsilon h} + \varphi_t^{\varepsilon h}, \end{aligned}$$

where $\phi_t^{\varepsilon h}$ is defined before and

$$\varphi_t^{\varepsilon h} = \int_0^t \left(\int_0^1 \nabla b(X_s + \nu(X_s^{\varepsilon h} - X_s), \alpha_s) d\nu - \nabla b(X_s, \alpha_s) \right) \psi_s^h ds.$$

By the condition **(H2)** (This is the only place we use this stronger condition rather than **(H1)** in the proof of the theorem), we obtain

$$\begin{aligned} \mathbb{E} \left[\sup_{s \leq t} \left| \frac{X_s^{\varepsilon h} - X_s}{\varepsilon} - \psi_s^h \right|^p \right] &\leq C \int_0^t \mathbb{E} \left| \frac{X_s^{\varepsilon h} - X_s}{\varepsilon} - \psi_s^h \right|^p ds + C \mathbb{E} \sup_{s \leq t} |\phi_s^{\varepsilon h}|^p \\ &\quad + C \left(\mathbb{E} \sup_{s \leq t} |X_s^{\varepsilon h} - X_s|^{2p} \right)^{1/2} \left(\mathbb{E} \sup_{s \leq t} |\psi_s^h|^{2p} \right)^{1/2}. \end{aligned}$$

By Lemmas 3.2–3.3, and Gronwall's inequality, we obtain

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\sup_{s \leq t} \left| \frac{X_s^{\varepsilon h} - X_s}{\varepsilon} - \psi_s^h \right|^p \right] = 0.$$

This implies that for $p \geq 2$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}_1 \left\| \frac{X_t^{\varepsilon h} - X_t}{\varepsilon} - \psi_t^h \right\|_{\mathbb{R}^n \otimes V}^p = 0.$$

Now, let $D_s X_t$ be the solution of the following equation:

$$D_s X_t = \sigma + \int_0^t \nabla b(X_r, \alpha_r) D_s X_r dr \quad s \leq S_t,$$

and $D_s X_t = 0$ for $s > S_t$. Then we can easily obtain that $\langle DX_t, h \rangle_H = \psi_t^h$ and $DX_t \in L^{\infty-}(\Omega_1, H \otimes \mathbb{R}^n \otimes V)$. The proof is complete. \clubsuit

Remark 3.1 *Following the same idea as the above we can prove that if the function $b(x, i)$ is infinitely differentiable in x with bounded partial derivative of all orders, then $X_t \in \mathbb{D}^\infty(\mathbb{R}^n \otimes V)$.*

Remark 3.2 *Theorem 3.1 says that $D^h X_t$ is a directional derivative of the solution X_t when we shift the Brownian motion with the deterministic function h . It is surprising to see that the switching variable α_t is not modified under this perturbation.*

The following theorem is the chain rule. Because the proof is almost the same as Theorem 3.6 in [4], we omit it.

Theorem 3.2 (Chain rule) *Assume that condition **(H2)** holds. Then for any $h \in H$, $p \geq 1$, $f \in C_b^2(\mathbb{R}^n \times \mathbb{S})$, we have*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left| \frac{f(X_t^{\varepsilon h}, \alpha_t^{\varepsilon h}) - f(X_t, \alpha_t)}{\varepsilon} - \nabla f(X_t, \alpha_t) D^h X_t \right|^p = 0.$$

Moreover, $f(X_t, \alpha_t) \in \mathbb{D}^{1,\infty}(V)$ and $Df(X_t, \alpha_t) = \nabla f(X_t, \alpha_t) DX_t$.

Definition 3.1 *Suppose that $F(x, \alpha) : \Omega \rightarrow \mathbb{R}^n$ is a measurable function for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{S}$. We say that its gradient with respect to x exists (in mean square sense) if there is $A(x, \alpha) : \Omega \rightarrow \mathbb{R}^{n^2}$ such that for any $\xi \in \mathbb{R}^n$ such that*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left| \frac{F(x + \varepsilon \xi, \alpha) - F(x, \alpha)}{\varepsilon} - A(x, \alpha) \xi \right|^2 = 0.$$

We denote the gradient matrix $A(x, \alpha)$ by $\nabla F(x, \alpha)$.

By an argument similar to that used in the proof of Theorem 3.1 or Theorem 4.2 in [12], we can obtain the following theorem.

Theorem 3.3 *Assume condition **(H2)** holds. Let $\{(X_t(x), \alpha_t(\alpha)), t \geq 0\}$ be the solution of (1.1) and (1.2), which starts from (x, α) . Then the gradient of $X_t(x)$ with respect to x (in mean square) exists. If we denote*

$$J_t := \nabla X_t(x),$$

then

$$J_t = I + \int_0^t \nabla b(X_s, \alpha_s) J_s ds, \quad (3.3)$$

where I is the n dimensional identity matrix. Moreover, J_t is invertible and its inverse K_t satisfies

$$K_t = I - \int_0^t K_s \nabla b(X_s, \alpha_s) ds. \quad (3.4)$$

By Gronwall's inequality, we can easily get $\max\{\|J_t\|, \|K_t\|\} \leq e^{\|\nabla b\|_\infty t}$. Using the integration by parts formula, we have

$$K_t D^h X_t = \int_0^t K_s \sigma d \left(\int_0^{S_s} h_r dr \right).$$

The Malliavin covariance matrix M_t is defined by:

$$M_t := \langle DX_t, (DX_t)^* \rangle_H.$$

By the method developed in Theorem 3.3 in [5], we can obtain that

$$M_t = J_t \int_0^t K_s \sigma \sigma^* K_s^* dS_s J_t^*$$

where J_t^* , σ^* and K_s^* are the matrix transposes of J_t , σ and K_s respectively.

4 Smoothness of density

In this section, we will prove that the solution X_t has a smooth density. As is well-known, the key step is to prove the Malliavin covariance matrix M_t has all the negative moments. The difficulty in our current situation is that b depends on the switching process α_t . Following the idea in [3] or [4], for any fixed $t > 0$, define $N_t := N([0, t], m_0(m_0 - 1)K)$, so N_t is a Poisson process with parameter $m_0(m_0 - 1)K$, conditioned on the number of jumps of the Poisson process up to time t , that is, $N_t = k$, there exists a random interval $[T_1, T_2]$ with $0 \leq T_1 < T_2 \leq t$ such that $T_2 - T_1 \geq \frac{t}{k+1}$ and $\alpha_t = \alpha_{T_1}$ for all $t \in [T_1, T_2]$ (because that the jump times of α_t is a subsequence of the jump times of N_t). On this time interval, we apply the results of Malliavin calculus in [14].

We first assume the following condition:

(H3) The Lévy measure ν_S satisfies for some $\theta \in (0, 2)$,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{\theta}{2}-1} \int_0^\varepsilon u \nu_S(du) = c_\theta > 0. \quad (4.1)$$

Remark 4.1 Let $\nu_S(du) = u^{-(1+\alpha/2)} du$ be the Lévy measure of $\alpha/2$ -stable subordinator. It is easy to see that (4.1) holds for $\theta = \alpha$.

The following lemma is a key step to prove that the Malliavin covariance matrix M_t has finite negative moments, which is a consequence of using Norris' type lemma. As is known to all, the classical Norris' lemma (for example, see Lemma 2.3.2 in [8]) is used to the case that the solution is a continuous semimartingale. However, the solution X_t here is a jump process, for this reason, we have to mention a recent paper [14], in which Zhang developed the Norris' type lemma for jump process. For our purpose in this paper, we will simplify this kind of Norris' type lemma and prove it also holds on time interval.

Lemma 4.1 Assume that condition **(H3)** holds. Let $V(x, i) : \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ be a function of infinitely differentiable in x with bounded partial derivative of all orders. For any $i \in \mathbb{S}$, $p \geq 2$, $\beta \in (0 \vee (4\theta - 7), 1)$, $0 \leq t_1 < t_2 \leq 1$, there exists $\varepsilon_0 = (t_2 - t_1)^{C(\beta, p)} \varepsilon(p)$, where $C(\beta, p)$ and $\varepsilon(p)$ are two positive constants dependent on β, p and p respectively, such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\sup_{|v|=1} \mathbb{P} \left(\int_{t_1}^{t_2} |v^* K_s^{(i)} [b, V](X_s^{(i)}, i)|^2 ds \geq \varepsilon^{\frac{1-\beta}{18-\beta}}, \int_{t_1}^{t_2} |v^* K_s^{(i)} V(X_s^{(i)}, i)|^2 ds \leq \varepsilon \right) \leq \varepsilon^p, \quad (4.2)$$

where $[b, V](x, i) = b(x, i) \cdot \nabla V(x, i) - V(x, i) \cdot \nabla b(x, i)$, $X_t^{(i)}$ and $K_t^{(i)}$ satisfy the following two equations respectively:

$$X_t^{(i)} = X_{t_1} + \int_{t_1}^t b(X_r^{(i)}, i) dr + \sigma(L_t - L_{t_1}), \quad t_1 \leq t \leq t_2$$

and

$$K_t^{(i)} = K_{t_1} - \int_{t_1}^t K_r^{(i)} \nabla b(X_r^{(i)}, i) dr, \quad t_1 \leq t \leq t_2.$$

Proof We first show that for any fixed $i \in \mathbb{S}$, $\beta \in (0 \vee (4\alpha - 7), 1)$, $0 \leq t_1 < t_2 \leq 1$, there exist two constants $C_1 \geq 1$ and $C_2 \in (0, 1)$ such that for all $\delta \in (0, 1)$,

$$\begin{aligned} & \sup_{|v|=1} \mathbb{P} \left(\int_{t_1}^{t_2} |v^* K_s^{(i)} [b, V](X_s^{(i)}, i)|^2 ds \geq (t_2 - t_1) \delta^{\frac{1-\beta}{2}}, \int_{t_1}^{t_2} |v^* K_s^{(i)} V(X_s^{(i)}, i)|^2 ds \leq (t_2 - t_1) \delta^{\frac{9-\beta}{2}} \right) \\ & \leq C_1 \exp\{-C_2(t_2 - t_1) \delta^{-\frac{\beta}{2}}\}. \end{aligned} \quad (4.3)$$

In fact, by changing of variable, we have that, for any $v \in \mathbb{R}^n$,

$$\begin{aligned} & \mathbb{P} \left(\int_{t_1}^{t_2} |v^* K_s^{(i)} [b, V](X_s^{(i)}, i)|^2 ds \geq (t_2 - t_1) \delta^{\frac{1-\beta}{2}}, \int_{t_1}^{t_2} |v^* K_s^{(i)} V(X_s^{(i)}, i)|^2 ds \leq (t_2 - t_1) \delta^{\frac{9-\beta}{2}} \right) \\ & = \mathbb{P} \left(\int_0^{t_2-t_1} |v^* \tilde{K}_s^{(i)} [b, V](\tilde{X}_s^{(i)}, i)|^2 ds \geq (t_2 - t_1) \delta^{\frac{1-\beta}{2}}, \int_0^{t_2-t_1} |v^* \tilde{K}_s^{(i)} V(\tilde{X}_s^{(i)}, i)|^2 ds \leq (t_2 - t_1) \delta^{\frac{9-\beta}{2}} \right), \end{aligned} \quad (4.4)$$

where $\tilde{K}_s^{(i)} := K_{t_1+s}^{(i)}$, $\tilde{X}_s^{(i)} := X_{t_1+s}^{(i)}$, for $0 \leq s \leq t_2 - t_1$. Obviously,

$$\tilde{X}_s^{(i)} = X_{t_1} + \int_0^s b(\tilde{X}_r^{(i)}, i) dr + \sigma \tilde{L}_s, \quad 0 \leq s \leq t_2 - t_1$$

and

$$\tilde{K}_s^{(i)} = K_{t_1} - \int_0^s \tilde{K}_r^{(i)} \nabla b(\tilde{X}_r^{(i)}, i) dr, \quad 0 \leq s \leq t_2 - t_1,$$

where $\tilde{L}_s := L_{t_1+s} - L_{t_1}$ is also a Lévy process and has the same distribution of L_s .

So, the estimate of (4.4) is changed into an estimate in the case of general SDE driven by subordinated Brownian motions without switching. Hence, applying Lemma 5.1 in [14] directly, it is easy to see that (4.3) holds.

Now, setting $\varepsilon := (t_2 - t_1) \delta^{9-\frac{\beta}{2}}$. Then notice that $\varepsilon^{\frac{1-\beta}{18-\beta}} \geq (t_2 - t_1) \delta^{\frac{1-\beta}{2}}$, and by (4.3), for any $\varepsilon \in (0, t_2 - t_1)$,

$$\begin{aligned} & \sup_{|v|=1} \mathbb{P} \left(\int_{t_1}^{t_2} |v^* K_s^{(i)} [b, V](X_s^{(i)}, i)|^2 ds \geq \varepsilon^{\frac{1-\beta}{18-\beta}}, \int_{t_1}^{t_2} |v^* K_s^{(i)} V(X_s^{(i)}, i)|^2 ds \leq \varepsilon \right) \\ & \leq C_1 \exp\{-C_2(t_2 - t_1)^{\frac{18}{18-\beta}} \varepsilon^{-\frac{\beta}{18-\beta}}\}. \end{aligned}$$

Therefore, there exists $\varepsilon_0 = (t_2 - t_1)^{C(\beta, p)} \varepsilon(p)$, where $C(\beta, p)$ and $\varepsilon(p)$ are two positive constants dependent on β, p and p respectively, such that for all $\varepsilon \in (0, \varepsilon_0)$, (4.2) holds. The proof is complete. \clubsuit

Next, we are going to study the smoothness of density for X_t . For the reason of the technique, we will divide the proof into two subsections.

4.1 If S_t has finite moments of all orders

In this section, we suppose that S_t has finite moments of all orders and $b \in C^\infty(\mathbb{R}^n \times \mathbb{S})$ has bounded derivatives of all orders.

Lemma 4.2 *For any $m, k \in \mathbb{N}$ with $m + k \geq 1$ and $p \geq 1$, we have*

$$\sup_{(x, \alpha) \in \mathbb{R}^n \times \mathbb{S}} \mathbb{E} \left(\|D^m \nabla^k X_t(x, \alpha)\|_{\mathcal{H}^{\otimes m} \otimes \mathbb{R}^{kn}}^p \right) < +\infty. \quad (4.5)$$

Proof By Theorem 3.1 and $\|J_t\| \leq e^{\|\nabla b\|_\infty t}$, we know that (4.5) holds for $m + k = 1$. For general m and k , it follows by similar calculations and induction method. \clubsuit

We need the following assumption for the Malliavin covariance matrix has all negative moments.

(H4) (Uniform Hörmander's type condition) There exists some $j_0 \in \mathbb{N}_+$, such that

$$\inf_{(x, i) \in \mathbb{R}^n \times \mathbb{S}} \inf_{|v|=1} \sum_{j=1}^{j_0} |v^* B_j(x, i)|^2 =: \kappa_1 > 0, \quad (4.6)$$

where $B_1(x, i) = \sigma$ and $B_{j+1}(x, i) := [b, B_j](x, i)$ for $j \in \mathbb{N}_+$.

Theorem 4.1 *Assume that conditions **(H3)** and **(H4)** hold. Then the Malliavin matrix M_t is invertible \mathbb{P} -a.s. and $\det(M_t^{-1}) \in L^p(\Omega)$ for all $p \geq 2$, $t \in (0, 1]$.*

Proof We recall that $M_t = J_{0,t} Q_t J_{0,t}^*$, where $Q_t := \int_0^t K_s \sigma \sigma^* K_s^* dS_s$. It suffices to prove $\det(Q_t^{-1}) \in L^p(\Omega)$ for all $p \geq 2$.

Recall that $\{N_t = N([0, t], m_0(m_0 - 1)K)\}$ is a Poisson process with parameter $\lambda := m_0(m_0 - 1)K$. For a fixed $t > 0$, conditioned on $N_t = k$, there exists a random interval $[T_1, T_2]$ such that $T_2 - T_1 \geq \frac{t}{k+1}$ and $\alpha_t = \alpha_{T_1}$ for all $t \in [T_1, T_2]$.

By lemma 3.1 in [13] and for the given θ in (4.1), and using the fact that the Poisson process N_t is independent of W, S , for any $p \geq 2$, there exists an $\varepsilon_0 = \varepsilon_0(\theta, p) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} & \mathbb{P} \left\{ \int_{T_1}^{T_2} |v^* K_s \sigma|^2 dS_s \leq \varepsilon \mid N_t = k \right\} \\ & \leq \mathbb{P} \left\{ \int_{T_1}^{T_2} |v^* K_s \sigma|^2 dS_s \leq \varepsilon, \int_{T_1}^{T_2} |v^* K_s \sigma|^2 ds \geq \varepsilon^{\theta/4} \mid N_t = k \right\} \\ & \quad + \mathbb{P} \left\{ \int_{T_1}^{T_2} |v^* K_s \sigma|^2 ds < \varepsilon^{\theta/4} \mid N_t = k \right\} \\ & \leq \exp \left\{ 1 - \frac{1}{2\varepsilon^{1-\theta/4}} \int_0^{\varepsilon} u \nu_S(du) \right\} + \mathbb{P} \left\{ \int_{T_1}^{T_2} |v^* K_s \sigma|^2 ds < \varepsilon^{\theta/4} \mid N_t = k \right\} \\ & \leq \exp \{ 1 - \varepsilon^{-\theta/8} \} + \mathbb{P} \left\{ \int_{T_1}^{T_2} |v^* K_s \sigma|^2 ds < \varepsilon^{\theta/4} \mid N_t = k \right\} \\ & \leq \varepsilon^p + \mathbb{P} \left\{ \int_{T_1}^{T_2} |v^* K_s \sigma|^2 ds < \varepsilon^{\theta/4} \mid N_t = k \right\}. \end{aligned} \quad (4.7)$$

Now, for any fixed $\beta \in (0 \vee (4\theta - 7), 1)$, $j = 1, 2, \dots, j_0$, denote $m(j) = (\frac{18-\beta}{1-\beta})^{j-1}$ and define

$$E_j := \left\{ \int_{T_1}^{T_2} |v^* K_s B_j(X_s, \alpha_s)|^2 ds < \varepsilon^{\frac{m_j \theta}{4}} \right\}.$$

Clearly, $\{\int_{T_1}^{T_2} |v^* K_s \sigma|^2 ds < \varepsilon^{\theta/4}\} = E_1$. Consider the decomposition

$$E_1 \subseteq (E_1 \cap E_2^c) \cup (E_2 \cap E_3^c) \cup \dots \cup (E_{j_0-1} \cap E_{j_0}^c) \cup F,$$

where $F = E_1 \cap E_2 \cap \dots \cap E_{j_0}$. Then for any unit vector v we have

$$\begin{aligned} \mathbb{P} \left\{ \int_{T_1}^{T_2} |v^* K_s \sigma|^2 ds < \varepsilon^{\theta/4} \middle| N_t = k \right\} &= \mathbb{P}(E_1 | N_t = k) \\ &\leq \mathbb{P}(F | N_t = k) + \sum_{j=1}^{j_0-1} \mathbb{P}(E_j \cap E_{j+1}^c | N_t = k). \end{aligned} \quad (4.8)$$

We are going to estimate each term in the above sum. This will be done in two steps.

Step 1: First we claim that when ε is sufficiently small, the intersection of F and $\{N_t = k\}$ is empty. In fact, taking into account (4.6), on $N_t = k$, we have

$$\begin{aligned} F &\subset \left\{ \sum_{j=1}^{j_0} \int_{T_1}^{T_2} |v^* K_s B_j(X_s, \alpha_s)|^2 ds \leq j_0 \varepsilon^{\frac{m_{j_0} \theta}{4}} \right\} \\ &= \left\{ \sum_{j=1}^{j_0} \int_{T_1}^{T_2} \left(\frac{|v^* K_s B_j(X_s, \alpha_s)|}{|v^* K_s|} \right)^2 |v^* K_s|^2 ds \leq j_0 \varepsilon^{\frac{m_{j_0} \theta}{4}} \right\} \\ &\subset \left\{ \frac{\kappa_1 t}{(k+1)e^{2\|\nabla b\|_\infty}} \leq j_0 \varepsilon^{\frac{m_{j_0} \theta}{4}} \right\}, \end{aligned}$$

because that $|v^* K_s| \geq \frac{1}{\|J_{0,s}\|} \geq \frac{1}{e^{\|\nabla b\|_\infty s}}$. Thus $F \cap \{N_t = k\} = \emptyset$, provided $\varepsilon < \varepsilon_1 := \left(\frac{\kappa_1 t}{j_0 e^{2\|\nabla b\|_\infty} (k+1)} \right)^{\frac{4}{m(j_0)\theta}}$.

Step 2: We shall bound the second terms in (4.8). For any $j = 1, 2, \dots, j_0 - 1$, we have

$$\begin{aligned} \mathbb{P}(E_j \cap E_{j+1}^c | N_t = k) &= \mathbb{P} \left\{ \int_{T_1}^{T_2} |v^* K_s B_j(X_s, \alpha_s)|^2 ds < \varepsilon^{\frac{m_j \theta}{4}}, \right. \\ &\quad \left. \int_{T_1}^{T_2} |v^* K_s B_{j+1}(X_s, \alpha_s)|^2 ds \geq \varepsilon^{\frac{m_{j+1} \theta}{4}} \middle| N_t = k \right\} \\ &= \mathbb{P} \left\{ \int_{T_1}^{T_2} |v^* K_s^{(\alpha_{T_1})} B_j(X_s^{(\alpha_{T_1})}, \alpha_{T_1})|^2 ds \leq \left(\varepsilon^{\frac{m_{j+1} \theta}{4}} \right)^{\frac{1-\beta}{18-\beta}}, \right. \\ &\quad \left. \int_{T_1}^{T_2} |v^* K_s^{(\alpha_{T_1})} B_{j+1}(X_s^{(\alpha_{T_1})}, \alpha_{T_1})|^2 ds \geq \varepsilon^{\frac{m_{j+1} \theta}{4}} \middle| N_t = k \right\}. \end{aligned}$$

Recall that $T_2 - T_1 \geq \frac{t}{k+1}$ and that processes N_t and L_t are independent, by using Lemma 4.1, we obtain

$$\mathbb{P}(E_j \cap E_{j+1}^c | N_t = k) \leq \varepsilon^p, \quad (4.9)$$

for $0 < \varepsilon \leq \varepsilon_2 = (\frac{t}{k+1})^{C(p)} \varepsilon(p)$, where $C(p)$ and $\varepsilon(p)$ are two positive constants dependent on p .

Hence, by (4.7)-(4.9), we have

$$\mathbb{P}\{v^*Q_tv \leq \varepsilon | N_t = k\} \leq \mathbb{P}\left\{\int_{T_1}^{T_2} |v^*K_s\sigma|^2 dS_s \leq \varepsilon | N_t = k\right\} \leq \varepsilon^p$$

for $\varepsilon < \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$. Then, following the steps of Lemma 2.3.1 in [8], we can obtain that

$$\mathbb{P}\left\{\inf_{|v|=1} v^*Q_tv \leq \varepsilon | N_t = k\right\} \leq \varepsilon^p$$

for all $0 < \varepsilon \leq C_1(\frac{t}{k+1})^{C_2}$ and for all $p \geq 2$, where C_1, C_2 are two positive constants depending on p and n . By the fact that $\det(Q_t) \geq (\inf_{|v|=1} v^*Q_tv)^n$, we have

$$\begin{aligned} \mathbb{E}|\det(Q_t)|^{-p} &\leq \mathbb{E}\left(\inf_{|v|=1} v^*Q_tv\right)^{-np} \\ &\leq \sum_{k=0}^{\infty} \mathbb{P}(N_t = k) \mathbb{E}\left(\left(\inf_{|v|=1} v^*Q_tv\right)^{-np} | N_t = k\right) \\ &\leq \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^\lambda \left[C_1 \left(\frac{t}{k+1}\right)^{C_2} + \frac{1}{C_1} \left(\frac{k+1}{t}\right)^{C_2} \right] < \infty. \end{aligned} \quad (4.10)$$

The proof is now complete. ♣

Now we can prove the following gradient estimate.

Theorem 4.2 *For any $k, m \in \mathbb{N}$ with $k + m \geq 1$, there are $\gamma_{k,m} > 0$ and $C = C_{k,m} > 0$ such that for all $f \in C_b^\infty(\mathbb{R}^n \times \mathbb{S})$ and $t \in (0, 1)$,*

$$|\nabla^k \mathbb{E}(\nabla^m f)(X_t, \alpha_t)| \leq C \|f\|_\infty t^{-\gamma_{k,m}}. \quad (4.11)$$

Proof By the chain rule, we have

$$\nabla^k \mathbb{E}(\nabla^m f)(X_t, \alpha_t) = \sum_{j=1}^k \mathbb{E}((\nabla^{m+j} f)(X_t, \alpha_t) G_j(\nabla X_t, \dots, \nabla^k X_t))$$

where $\{G_j, j = 1, \dots, k\}$ are real polynomial functions. By duality relationship, chain rule, Lemma 4.2 and Hölder's inequality, through cumbersome calculation (the details follow the argument of Proposition 2.1.4 in [8]), one finds that there exist integer $p = p_{k,m}$, $C = C_{k,m} > 0$ and $\gamma_{k,m} > 0$ such that for all $t \in (0, 1)$,

$$|\nabla^k \mathbb{E}(\nabla^m f)(X_t, \alpha_t)| \leq C \|f\|_\infty \mathbb{E}|\det M_t|^{-1/p} \leq C \|f\|_\infty t^{-\gamma_{k,m}},$$

where the last inequality follows by (4.10). The proof is complete. ♣

4.2 Without the assumption of S_t has finite moments of all orders.

Let S'_t be a subordinator with Lévy measure $1_{(0,1)}\nu_S(du)$ and independent of $(W_t)_{t \geq 0}$ and $N(dt, dy)$. Let (X'_t, α'_t) solves the following equations:

$$dX'_t = b(X'_t, \alpha'_t)dt + \sigma dW_{S'_t}, \quad (X'_0, \alpha'_0) = (x, \alpha) \in \mathbb{R}^n \times \mathbb{S}, \quad (4.12)$$

and

$$\mathbb{P}\{\alpha'_{t+\Delta} = j | \alpha'_t = i, X'_s, \alpha'_s, s \leq t\} = \begin{cases} q_{ij}(X'_t)\Delta + o(\Delta), & i \neq j \\ 1 + q_{ii}(X'_t)\Delta + o(\Delta), & i = j. \end{cases} \quad (4.13)$$

Let us write

$$P'_t f(x, \alpha) := \mathbb{E} f(X'_t(x), \alpha'_t(\alpha)).$$

Notice that S'_t has finite moments of all order, the results above also hold for the process (X'_t, α'_t) . Hence, we intend to find the relation between the semigroups P_t and P'_t , so that we can estimate the semigroup P_t through by estimating P'_t . Following the steps of subsection 3.3 in [13], we first give out two lemmas whose proofs are almost the same with Lemmas 3.9 and 3.10 in [13], so we omit the proof.

Lemma 4.3 *Let $f \in C_b^\infty(\mathbb{R}^n \times \mathbb{S})$. For any $m \in \mathbb{N}$, there exists a constant $C_m \geq 1$ such that for all $(x, \alpha) \in \mathbb{R}^n \times \mathbb{S}$,*

$$|\nabla^m P'_t f(x, \alpha)| \leq C_m \sum_{k=1}^m P'_t |\nabla^k f|(x, \alpha). \quad (4.14)$$

Lemma 4.4 *Let $J'_t := \nabla X'_t(x)$ and K'_t be the inverse of matrix J'_t . Let $f \in C_b^\infty(\mathbb{R}^n \times \mathbb{S})$. Then for any $j = 1, \dots, n$, we have the following formula:*

$$P'_t(\partial_j f)(x, \alpha) = \text{div} Q^j(t, x, \alpha; f) - G^j(t, x, \alpha; f), \quad (4.15)$$

where

$$Q^{ij}(t, x, \alpha; f) := \mathbb{E}(f(X'_t(x), \alpha'_t(\alpha))(K'_t)_{ij}) \quad (4.16)$$

and

$$G^j(t, x, \alpha; f) := \mathbb{E}(f(X'_t(x), \alpha'_t(\alpha)) \text{div}(K'_t)_j). \quad (4.17)$$

Now, let $\{\tau'_1, \tau'_2, \dots, \tau'_n, \dots\}$ and $\{\xi_1, \xi_2, \dots, \xi_n, \dots\}$ be two independent families of i.i.d. random variables in \mathbb{R}^+ and \mathbb{R}^d respectively, which are also independent of $(W_t, S'_t)_{t \geq 0}$ and $N(dt, du)$. We assume that τ'_1 has the expoential distribution of parameter $\lambda_1 := \nu_S([1, \infty))$ and ξ_1 has the distributional density

$$\frac{1}{\nu_S([1, \infty))} \int_1^\infty (2\pi s)^{-d/2} e^{-\frac{|x|^2}{2s}} \nu_S(ds).$$

Set $\tau'_0 := 0$ and $\xi_0 := 0$, define

$$N'_t := \max\{k : \tau'_0 + \tau'_1 + \dots + \tau'_k \leq t\} = \sum_{k=0}^{\infty} 1_{\{\tau'_0 + \tau'_1 + \dots + \tau'_k \leq t\}}$$

and

$$H_t := \xi_0 + \xi_1 + \dots + \xi_{N'_t} = \sum_{j=0}^{N'_t} \xi_j.$$

Then H_t is a compound Poisson process with Lévy measure

$$\nu_H(\Gamma) = \int_1^\infty (2\pi s)^{-d/2} \left(\int_\Gamma e^{-\frac{|y|^2}{2s}} dy \right) \nu_S(ds).$$

Moreover, it is easy to see that H_t is independent of $W_{S'_t}$ and

$$(\sigma W_{S_t})_{t \geq 0} \stackrel{(d)}{=} (\sigma W_{S'_t} + \sigma H_t)_{t \geq 0}. \quad (4.18)$$

Let \bar{h}_t be a càdlàg purely discontinuous \mathbb{R}^n -valued function with finite many jumps and $\bar{h}_0 = 0$. Let $(X_t^{\bar{h}}(x), \alpha_t^{\bar{h}}(\alpha))$ solve the following equations:

$$dX_t^{\bar{h}} = b(X_t^{\bar{h}}, \alpha_t^{\bar{h}})dt + \sigma dW_{S_t'} + d\bar{h}_t, \quad (X_0', \alpha_0') = (x, \alpha) \in \mathbb{R}^n \times \mathbb{S}, \quad (4.19)$$

and

$$\mathbb{P}\{\alpha_{t+\Delta}^{\bar{h}} = j | \alpha_t^{\bar{h}} = i, X_s^{\bar{h}}, \alpha_s^{\bar{h}}, s \leq t\} = \begin{cases} q_{ij}(X_t^{\bar{h}})\Delta + o(\Delta), & i \neq j \\ 1 + q_{ii}(X_t^{\bar{h}})\Delta + o(\Delta), & i = j. \end{cases} \quad (4.20)$$

Let k be the jump number of \bar{h} before time t . Let $0 = t_0 < t_1 < t_2 < \dots < t_k \leq t$ be the jump times of \bar{h} . By the Markovian property of $(X_t^{\bar{h}}(x), \alpha_t^{\bar{h}}(\alpha))$, we have the following formula:

$$\mathbb{E}f(X_t^{\bar{h}}(x), \alpha_t^{\bar{h}}(\alpha)) = P'_{t_1} \cdots \theta_{\Delta \bar{h}_{t_{k-1}}} P'_{t_k - t_{k-1}} \theta_{\Delta \bar{h}_{t_k}} P'_{t - t_k} f(x, \alpha),$$

where

$$\theta_y f(x, \alpha) := f(x + y, \alpha).$$

Now, by (4.18) we have

$$(X_t(x), \alpha_t(\alpha)) \stackrel{(d)}{=} (X_t^{\bar{h}}(x), \alpha_t^{\bar{h}}(\alpha))|_{\bar{h}=\sigma H}.$$

Hence,

$$\begin{aligned} P_t f(x, \alpha) &= \mathbb{E}f(X_t(x), \alpha_t(\alpha)) = \mathbb{E}(f(X_t^{\bar{h}}(x), \alpha_t^{\bar{h}}(\alpha))|_{\bar{h}=\sigma H}) \\ &= \sum_{k=0}^{\infty} \mathbb{E}\left(P'_{\tau_1} \cdots \theta_{\sigma \xi_{k-1}} P'_{\tau'_k} \theta_{\sigma \xi_k} P'_{t-(\tau'_1+\dots+\tau'_k)} f(x, \alpha), N'_t = k\right). \end{aligned}$$

In view of

$$\{N'_t = k\} = \{\tau'_1 + \dots + \tau'_k \leq t < \tau'_1 + \dots + \tau'_k + \tau'_{k+1}\},$$

we further have

$$\begin{aligned} P_t f(x, \alpha) &= \sum_{k=0}^{\infty} \left\{ \int_{\sum_{i=1}^k t_i \leq t < \sum_{i=1}^{k+1} t_i} \mathbb{E}(P'_{t_1} \cdots \theta_{\sigma \xi_{k-1}} P'_{t_k} \theta_{\sigma \xi_k} P'_{t-\sum_{i=1}^k t_i} f(x, \alpha)) \right. \\ &\quad \left. \times \lambda_1^{k+1} e^{-\lambda_1 \sum_{i=1}^{k+1} t_i} dt_1 \cdots dt_{k+1} \right\} + P'_t f(x, \alpha) \mathbb{P}(N'_t = 0) \\ &= \sum_{k=1}^{\infty} \left\{ \lambda_1^k e^{-\lambda_1 t} \int_{\sum_{i=1}^k t_i \leq t} \mathbb{E} I_f^{\sigma \xi}(t_1, \dots, t_k, t, x, \alpha) dt_1 \cdots dt_k \right\} \\ &\quad + P'_t f(x, \alpha) e^{-\lambda_1 t}, \end{aligned} \quad (4.21)$$

where $\xi := (\xi_1, \dots, \xi_k)$ and

$$I_f^{\mathbf{y}}(t_1, \dots, t_k, t, x, \alpha) := P'_{t_1} \cdots \theta_{y_{k-1}} P'_{t_k} \theta_{y_k} P'_{t-(t_1+\dots+t_k)} f(x, \alpha)$$

with $\mathbf{y} := (y_1, \dots, y_k)$.

With preparations above, we can give the main result.

Theorem 4.3 *Let $b \in C^\infty(\mathbb{R}^n \times \mathbb{S}; \mathbb{R}^n)$ with bounded partial derivatives of all orders. Suppose conditions **(H3)** and **(H4)** hold. Then X_t has a smooth density with respect to the Lebesgue measure on \mathbb{R}^n .*

Proof In order to prove the smoothness of density for X_t . By [7, 8], it suffices to show that for any $f \in C_b^\infty(\mathbb{R}^n)$, we have

$$|\mathbb{E} \nabla_{i_1, \dots, i_m}^m f(X_t)| \leq C \|f\|_\infty, \quad \forall m \geq 1, (i_1, \dots, i_m) \in \{1, \dots, n\}^m,$$

where $\nabla_{i_1, \dots, i_m}^m = \frac{\partial^m}{\partial x_{i_1} \dots \partial x_{i_m}}$ and C depends on $t, x, (i_1, \dots, i_m)$. However, this can be easily obtained if we can establish the same gradient estimate as in (4.11).

If we let $t_{k+1} := t - (t_1 + \dots + t_k)$, then there exists at least one $j \in \{1, 2, \dots, k+1\}$ such that $t_j \geq \frac{t}{k+1}$. Thus, by (4.14) and (4.11), we have

$$\begin{aligned} |\nabla I_f^y(t_1, \dots, t_k, t, x, \alpha)| &\leq C_1^{j-1} \|\nabla P'_{t_j} \dots \theta_{y_{k-1}} P'_{t_k} \theta_{y_k} P'_{t_{k+1}} f\|_\infty \\ &\leq C C_1^{j-1} t_j^{-\gamma_{1,0}} \|P'_{t_{j+1}} \dots \theta_{y_{k-1}} P'_{t_k} \theta_{y_k} P'_{t_{k+1}} f\|_\infty \\ &\leq C C_1^k \left(\frac{t}{k+1}\right)^{-\gamma_{1,0}} \|f\|_\infty. \end{aligned}$$

Hence, by (4.21) we have

$$\begin{aligned} |\nabla P_t f(x, \alpha)| &\leq C \|f\|_\infty t^{-\gamma_{1,0}} e^{-\lambda_1 t} \left[1 + \sum_{k=1}^{\infty} \lambda_1^k C_1^k (k+1)^{\gamma_{1,0}} \int_{\sum_{i=1}^k t_i \leq t} dt_1 \dots dt_k \right] \\ &= C \|f\|_\infty t^{-\gamma_{1,0}} e^{-\lambda_1 t} \left(\sum_{k=0}^{\infty} \lambda_1^k C_1^k (k+1)^{\gamma_{1,0}} \frac{t^k}{k!} \right) \\ &\leq C \|f\|_\infty t^{-\gamma_{1,0}}. \end{aligned} \tag{4.22}$$

Thus, we obtain (4.11) with $k = 1$ and $m = 0$.

For $l, i = 1, \dots, n$, set $F_{li}^0(x, \alpha) := 1_{\{l=i\}} f(x, \alpha)$. Let us recursively define for $m = 0, 1, \dots, k$,

$$F_{li}^{(m+1)}(x, \alpha) := \sum_{j=1}^n Q^{ij}(t_{k+1-m}, x + y_k + \dots + y_{k-m}, \alpha; F_{lj}^{(m)})$$

and

$$R_l^{(m+1)}(x, \alpha) := \sum_{j=1}^n G^j(t_{k+1-m}, x + y_k + \dots + y_{k-m}, \alpha; F_{lj}^{(m)}),$$

where Q^{ij} and G^j are defined by (4.16) and (4.17). From these definitions, it easy to see that

$$\|F_{li}^{(m+1)}\|_\infty \leq \sum_{j=1}^n \|F_{lj}^{(m)}\|_\infty \mathbb{E}((K'_t)_{ij}) \leq C \sum_{j=1}^n \|F_{lj}^{(m)}\|_\infty \leq C n^m \|f\|_\infty$$

and

$$\|R_l^{(m+1)}\|_\infty \leq \sum_{j=1}^n \|F_{lj}^{(m)}\|_\infty \mathbb{E}((K'_t)_{ij}) \mathbb{E}(\text{div}(K'_t)_{\cdot j}) \leq C n^{m+1} \|f\|_\infty.$$

By repeatedly using Lemma 4.4, we obtain

$$\begin{aligned} &|I_{\partial_l f}^y(t_1, \dots, t_k, t, x, \alpha)| \\ &= \left| P'_{t_1} \dots \theta_{y_{j-1}} P'_{t_j} \text{div} F_{li}^{(k+1-j)}(x, \alpha) - \sum_{m=1}^{k+1-j} P'_{t_1} \dots \theta_{y_{k-m}} P'_{t_{k+1-m}} R_l^{(m)}(x, \alpha) \right| \\ &\leq C t_j^{-\gamma_{1,0}} \sum_{i=1}^n \|F_{li}^{(k+1-j)}\|_\infty + \sum_{m=1}^{k+1-j} \|R_l^{(m)}\|_\infty \\ &\leq C \left(\frac{t}{k+1}\right)^{-\gamma_{1,0}} \|f\|_\infty + C \|f\|_\infty. \end{aligned}$$

As estimating in (4.22), we can obtain (4.11) with $k = 0$ and $m = 1$. For the general m and k , the gradient estimate (4.11) follows by similar calculations and induction method. The proof is complete. ♣

References

- [1] Z. Dong and Y. Xie, Yingchao, *Ergodicity of stochastic 2D Navier-Stokes equation with Lévy noise*, J. Differential Equations 251 (2011) 196-222.
- [2] Z. Dong and Y. Xie, Yingchao, *Global solutions of stochastic 2D Navier-Stokes equations with Lévy noise*, Sci. China Ser. A 52 (2009) 1497-1524.
- [3] B. Forster, E. Lütkebohmert and J. Teichmann, *Absolutely continuous laws of jump-diffusions in finite and infinite dimensions with application to mathematical finance*, SIAM J. Math. Anal. 40(5) (2009) 2132-2153.
- [4] Y. Hu, D. Nualart, X. Sun and Y. Xie, *smoothness of density and ergodicity for state-dependent switching diffusion*. arXiv:1409.3927v1.
- [5] S. Kusuoka, *Malliavin calculus for stochastic differential equations driven by subordinated Brownian motions*, Kyoto J. of Math. 50(3) (2009), 491-520.
- [6] P. Malliavin, *Stochastic analysis*, Springer-Verlag, Berlin, 1997.
- [7] J. Norris, *Simplified Malliavin calculus*, Seminaire de probalilities (Strasbourg), 20 (1986), 101-130.
- [8] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, Berlin, 2006.
- [9] F. Xi and G. Yin, *Jump-diffusions with state-dependent switching: existence and uniqueness, Feller property, linearization, and uniform ergodicity*, Science in China Series A: Mathematics, 54(12) (2011), 2651C2667.
- [10] F. Xi and G. Yin, *The strong Feller property of switching jump-diffusion processes*, Stat Probab Letters, 83 (2013) 761-767.
- [11] Y. Xie, *On the estimations of smooth densities for integro-differential operators*, Stochastic Anal. Appl. 22 (2004) 211-236.
- [12] G. Yin and C. Zhu, *Properties of solutions of stochastic differential equations with continuous-state-dependent switching*, J. Differential Equations 249 (2012) 2409-2439.
- [13] X. Zhang, *Densities for SDEs driven by degenerate α -stable processes*, Ann. Probab. 42 (2014), 1885-1910.
- [14] X. Zhang, *Nonlocal Hörmander's hypoellipticity theorem*, arXiv.1306.5016v4.